Unit - II

Linear Transformations

Sec: 6.1 The Algebra of linear transformations

Defn: A non-empty set R is said to be associative ring if in R there are defined two operation denoted by "+" and "." respectively such that for all a, b, c in R.

1) $a+b$ in R
2) $a+b = b+a$
3) $a+(b+c) = (a+b)+c$
4) there is an element $0 \in R$ such that $0+a = a+0 = a$
5) $\exists$ $-a \in R$ such that $-a+a = 0 = a+(-a)$
6) $a \cdot b \in R$
7) $a\cdot(b\cdot c) = (a\cdot b)\cdot c$
8) $a\cdot(b+c) = a\cdot b + a\cdot c$ and $(a+b)\cdot c = a\cdot c + b\cdot c$

Defn: An associative ring A is said to be an algebra over F if A is a vector space over F such that for all $a, b \in A$ and $\alpha \in F$. $\alpha(ab) = (\alpha a)b = a(\alpha b)$.

Defn: The element of A(V) is called the linear transformation on V over F. Hence A(V) is called algebra of linear transformation on V.

1-7-2009 Lemma: 6.1.1

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[Analog of Cayley's Theorem for algebra]

AP-2019 If A is an algebra, with unit element, over F, then A is isomorphic to a subalgebra of A(V) for some vector space V over F.

Proof:

Since A is an algebra over F, A is a vector space over F.

We shall use $V = A$

If $a \in A$ define $T_a : A \to A$

ie) $T_a : V \to V$ by $vT_a = va$ for every $v \in A$.

<u>To prove</u>: T_a is a linear transformation on V.

~~ie) To prove: $T_a \in A(V)$~~.

Let $v_1, v_2 \in V$ and $\alpha \in F$.

Then
$$\begin{aligned}(v_1 + v_2) T_a &= (v_1 + v_2) a \\ &= v_1 a + v_2 a \\ &= v_1 T_a + v_2 T_a\end{aligned}$$

$$\begin{aligned}(\alpha v_1) T_a &= (\alpha v_1) a \\ &= \alpha (v_1 a) \\ &= \alpha (v_1 T_a)\end{aligned}$$

$\therefore$ T_a is a linear transformation on V.

$\therefore$ $T_a \in A(V)$.

Define $\psi : A \to A(V)$ by $a\psi = T_a \; \forall \, a \in A$

<u>To Prove</u>: ψ is an isomorphism from A into $A(V)$.

Let $a, b \in A$, $\alpha, \beta \in F$ and $T_a, T_b \in A(V)$.

<u>To prove</u>: $(\alpha a + \beta b)\psi = \alpha(a\psi) + \beta(b\psi)$

For any $v \in V$.

$$\begin{aligned}v(T_{\alpha a + \beta b}) &= v(\alpha a + \beta b) \\ &= v(\alpha a) + v(\beta b) \\ &= \alpha(va) + \beta(vb) \\ &= \alpha(vT_a) + \beta(vT_b) \\ &= v(\alpha T_a) + v(\beta T_b) \\ &= v(\alpha T_a + \beta T_b) \text{ is true for all } v \in V\end{aligned}$$

$$T_{\alpha a + \beta b} = \alpha T_a + \beta T_b.$$

Now $(\alpha a + \beta b)\psi = T_{\alpha a + \beta b}$.

$= \alpha T_a + \beta T_b$

$= \alpha(a\psi) + \beta(b\psi)$

$(\alpha a + \beta b)\psi = \alpha(a\psi) + \beta(b\psi)$

$\therefore \psi$ is a vectorspace homomorphism of A into A(V).

<u>claim:</u> $T_{ab} = T_a \cdot T_b \quad \forall \; T_a, T_b \in A(V)$.

Let $v \in V$.

$v(T_{ab}) = v(ab)$

$= (va)b$

$= (vT_a)T_b$

$= v(T_a T_b)$

ie) $v(T_{ab}) = v(T_a \cdot T_b)$ is true for all $v \in V$.

$\Rightarrow T_{ab} = T_a \cdot T_b$.

Now $(ab)\psi = T_{ab}$

$= T_a T_b$.

$(ab)\psi = a\psi \cdot b\psi$

~~ψ is also a ring homomorphism of A~~

<u>To prove:</u> ψ is 1-1

ie) <u>To prove:</u> $\ker\psi = \{0\}$.

Let $a \in \ker\psi$.

Then $a\psi = 0$.

$\Rightarrow T_a = 0$.

$\Rightarrow vT_a = 0 \; \forall \; v \in V = A$.

$\Rightarrow eT_a = 0$. [$\because$ A is an algebra with unit element e]

$\Rightarrow e.a = 0$.

Hence ψ is an isomorphism from A into A(V).

$\therefore$ A is isomorphic to the image of A under ψ.

ie) Range of ψ which is subalgebra of A(V).

Note: Let A be an algebra with unit element e over and let $P(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \cdots + \alpha_n x^n$ be a polynomial in $F[x]$.

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Lemma: 6.1.2

Let A be an algebra, with unit element, over F, and suppose that A is of dimension m over F. Then every element in A satisfies some nontrivial polynomial in $F[x]$ of degree atmost m.

Proof:

Let A be an algebra over F with unit element e and $\dim_F A = m$.

Let $a \in A$

Consider the $m+1$ elements $e, a, a^2, \ldots, a^m$ in A.

since $\dim_F A = m$, these $(m+1)$ elements are linearly dependent over F [By Lemma: 4.2.4]

ie) $\exists$ some elements $\alpha_0, \alpha_1, \ldots, \alpha_m$ not all zero such that $\alpha_0 e + \alpha_1 a + \cdots + \alpha_m a^m = 0$.

$\Rightarrow$ a satisfies a nontrivial polynomial.

$P(x) = \alpha_0 + \alpha_1 x + \cdots + \alpha_m x^m$ of degree atmost m.

Thus every element in A satisfies some nontrivial polynomial in $F[x]$ of degree atmost m.

Theorem: 6.1.1

If V is an n-dimensional vector space over F, then given by any element T in $A(V)$, there exist a nontrivial polynomial $q(x) \in F[x]$ of degree atmost n^2, such that $q(T) = 0$.

Proof:

Let V be an n-dimensional vector space over F.

Then $A(V)$ is an algebra over F with unit element and $\dim_F A(V) = n^2$. [$\because \dim_F V = n$ then $\dim_F \mathrm{Hom}(V,V) = n^2$] (cor) 10 Remark 3)

write the Lemma

By the above Lemma, any element in $A(V)$ satisfies a nontrivial polynomial $q(x)$ in $F[x]$ of degree atmost n^2 such that $q(T) = 0$.

Defn: If V is finite dimensional over F and $T \in A(V)$. Then a nontrivial polynomial $p(x)$ in $F(x)$ of smallest positive degree satisfied by T is called minimal polynomial of T over F.

Note: If $p(x)$ is a minimal polynomial of T over F and $q(x) \in F[x]$ is satisfied by T then $p(x) / q(x)$.

Defn: (i) An element $T \in A(V)$ is called right invertible if $\exists$ an element $S \in A(V)$ such that $TS = 1$.

(ii) An element $T \in A(V)$ is called left invertible if $\exists$ an element $U \in A(V)$ such that $UT = 1$.

Remark: If T is both right and left invertible and if $TS = UT = 1$ then $S = U$ and S is unique.

Defn: An element $T \in A(V)$ is called invertible or regular if there exist an element $S \in A(V)$ such that $ST = TS = 1$.

ie) It is both left and right invertible.

Here we write S as T^{-1}

Defn: An element $T \in A(V)$ is called singular if it is not regular.

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~~Example~~ Note: 1. An element T in $A(V)$ is invertible or regular if it is ~~bot~~ which is not regular is called singular 2. An elt in $A(V)$ is right invertible but is not invertible

Let F be the field of real numbers and $V = F[x]$ the set of all polynomial in x over F.

Define S on V by $q(x) \cdot S = \frac{d}{dx} q(x)$.

Then $S \in A(V)$.

Define T on V by $q(x) T = \int_1^x q(x) dx$.

Then $ST \neq 1$, Then $TS = 1$.

For
$$q(x) TS = (q(x)T)S$$
$$= \left(\int_1^x q(x)\, dx \right) S$$
$$= \frac{d}{dx} \int_1^x q(x)\, dx = q(x).$$

$$TS = 1.$$

$\therefore$ T is right invertible.

Now,
$$q(x) ST = (q(x) S) T$$
$$= \left(\frac{d}{d(x)} q(x) \right) T$$
$$= \int_1^x \frac{d}{dx} q(x)\, dx$$
$$= \left(q(x) \right)_1^x$$
$$= q(x) - q(1) \neq q(x)$$

$$q(x) ST \neq q(x).$$

$$\Rightarrow ST \neq 1$$

$\Rightarrow$ T is not left invertible.

2018 J ~~Lemma: 6.1.2~~ Theorem: 6.1.2

If V is finite - dimensional over F, then $T \in A(V)$ is invertible iff the constant term of the minimal polynomial T is not 0.

Proof:

Let $p(x) = \alpha_0 + \alpha_1 x + \cdots + \alpha_k x^k$ where $\alpha_k \neq 0$. be the minimal polynomial for T in $F[x]$.

Assume that $\alpha_0 \neq 0$.

To prove: $T \in A(V)$ is invertible.

Since $p(T) = 0 \Rightarrow \alpha_0 + \alpha_1 T + \cdots + \alpha_k T^k = 0$.

$\Rightarrow \alpha_1 T + \alpha_2 T^2 + \cdots + \alpha_k T^k = -\alpha_0$.

$\div(-\alpha_0) \Rightarrow \frac{-1}{\alpha_0}\left(\alpha_1 + \alpha_2 T + \cdots + \alpha_k T^{k-1}\right) T = 1$

ie) $T\left[\frac{-1}{\alpha_0}\left(\alpha_1 + \alpha_2 T + \cdots + \alpha_k T^{k-1}\right)\right] = 1$.

Take $S = \frac{-1}{\alpha_0}\left(\alpha_1 + \alpha_2 T + \cdots + \alpha_k T^{k-1}\right)$

$\therefore ST = TS = 1$.

Hence T is invertible.

Conversely, Assume T is invertible.

To prove: $\alpha_0 \neq 0$.

Suppose $\alpha_0 = 0$.

$p(T) = 0 \Rightarrow \alpha_1 T + \alpha_2 T^2 + \cdots + \alpha_k T^k = 0$.

$\Rightarrow T(\alpha_1 + \alpha_2 T + \cdots + \alpha_k T^{k-1}) = 0$.

Since T is invertible $\Rightarrow T^{-1}$ exists.

Then $T^{-1} T(\alpha_1 + \alpha_2 T + \cdots + \alpha_k T^{k-1}) = T^{-1} 0$

$\alpha_1 + \alpha_2 T + \cdots + \alpha_k T^{k-1} = 0$.

Which is a contradiction to $p(x)$ is the minimal polynomial for T over F

Corollary: 1

If V is finite-dimensional over F and if $T \in A(V)$ is invertible, then T^{-1} is a polynomial expression in T over F.

Proof:

Let $P(x) = \alpha_0 + \alpha_1 x + \dots + \alpha_k x^k$ where $\alpha_k \neq 0$ be the minimal polynomial for T over F.

Suppose $T \in A(V)$ is invertible.

write the Theorem

Then by the above Theorem, $\alpha_0 \neq 0$.

Now, $P(T) = 0 \Rightarrow \alpha_0 + \alpha_1 T + \dots + \alpha_k T^k = 0$

$$\Rightarrow \alpha_1 T + \alpha_2 T^2 + \dots + \alpha_k T^k = -\alpha_0$$

$$\Rightarrow \frac{-1}{\alpha_0}\left(\alpha_1 + \alpha_2 T + \dots + \alpha_k T^{k-1}\right) T = 1.$$

ie) $T\left[\frac{-1}{\alpha_0}\left(\alpha_1 + \alpha_2 T + \dots + \alpha_k T^{k-1}\right)\right] = 1.$

Take $S = \frac{-1}{\alpha_0}\left(\alpha_1 + \alpha_2 T + \dots + \alpha_k T^{k-1}\right)$

$\therefore ST = TS = 1.$

$\Rightarrow S$ is the inverse of T (ie) $S = T^{-1}$

$$\therefore T^{-1} = \frac{-1}{\alpha_0}\left(\alpha_1 + \alpha_2 T + \dots + \alpha_k T^{k-1}\right)$$

$\Rightarrow T^{-1}$ is a polynomial expression in T over F.

Corollary: 2

If V is finite-dimensional over F and if $T \in A(V)$ is singular, then there exists an $S \neq 0$ in $A(V)$ such that $ST = TS = 0$.

Proof:

Let $P(x) = \alpha_0 + \alpha_1 x + \dots + \alpha_k T^k$ where $\alpha_k \neq 0$ be the minimal polynomial for T over F.

write the Theorem

Since $T \in A(V)$ is singular.

By Theorem: 6.1.2, the constant term of minimal polynomial α_0 is zero.

$$\therefore p(x) = \alpha_1 x + \cdots + \alpha_k T^k$$

Then $p(T) = 0 \Rightarrow \alpha_1 T + \alpha_2 T^2 + \cdots + \alpha_k T^k = 0$.

$$(\alpha_1 + \alpha_2 T + \cdots + \alpha_k T^{k-1}) T = 0$$

ie) $T(\alpha_1 + \alpha_2 T + \cdots + \alpha_k T^{k-1}) = 0$.

Take $S = \alpha_1 + \alpha_2 T + \cdots + \alpha_k T^{k-1}$.

Then $ST = TS = 0$ and $S \in A(V)$.

claim: $S \neq 0$.

Suppose $S = 0$.

$\Rightarrow \alpha_1 + \alpha_2 T + \cdots + \alpha_k T^{k-1} = 0$

$\Rightarrow$ T satisfies a polynomial of degree $k-1$

Which is a contradiction to $p(x)$ is a minimal polynomial for T.

$\therefore S \neq 0$

Hence there exist $S \neq 0$ in $A(V)$ such that $ST = TS = 0$.

corollary: 3

If V is finite-dimensional over F and if $T \in A(V)$ is right invertible. Then it is invertible.

Proof:

Let $T \in A(V)$ is right invertible.

Then there exist $U \in A(V)$ such that $TU = 1$.

<u>To prove:</u> T is invertible.

Suppose T is not invertible.

write the corollary ie) T is singular. Then by above corollary, there exists $S \neq 0$ in $A(V)$ such that $ST = TS = 0$.

Now, $S = S \cdot 1$

$= S(TU) = (ST)U$

$= 0.U = 0.$

$\Rightarrow S = 0.$

Which is a contradiction to $S \neq 0$.

$\therefore$ T is invertible.

Theorem: 6.1.3

If V is finite dimensional over F, then $T \in A(V)$ is singular iff there exists a $v \neq 0$ in V such that $vT = 0$.

Proof:

Suppose $T \in A(V)$ is singular. then there exist $S \neq 0$ in $A(V)$ such that $ST = TS = 0$.

Since $S \neq 0$.

Then there exists $w \in V$ such that $wS \neq 0$ in V

Let $v = wS$

$\Rightarrow v \neq 0$ in V.

Now, $vT = (wS)T$

$= w(ST)$

$= w(0)$

$= 0.$

$\therefore vT = 0.$

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Hence there exists an element $v \neq 0$ in V Such that $vT = 0$.

Conversely, $vT = 0$ with $v \neq 0$.

<u>To Prove</u>: T is singular

Suppose T is not singular.

ie) T is invertible, then there exists $S \in A(V)$ Such that $ST = TS = 1$.

Here $vT = 0$.

$\Rightarrow (vT)S = 0S$

$\Rightarrow (vT)\ v(TS) = 0.$

$\Rightarrow v \cdot 1 = 0$, (ie) $v = 0$.

which is a contradiction to our assumption $v \neq 0$.

$\therefore$ T is singular.

Defn: If $T \in A(V)$ then the range of T is denoted by VT and is defined by $VT = \{vT / v \in V\}$

Note:

(i) Range of T is a subspace of V.

(ii) Range of T is all of V iff T is onto.

Theorem: 6.1.4

2009, 2014 J

If V is finite dimensional over F, then $T \in A(V)$ is regular iff T maps V onto V.

Proof:

Assume T is regular.

Then there exists $T^{-1} \in A(V)$ such that $TT^{-1} = T^{-1}T = 1$

Let $v \in V$.

Now, $v = v \cdot 1$

$= v(T^{-1}T)$

$= (vT^{-1})T \in VT$.

Also $VT \subseteq V$

$\therefore VT = V$.

By Note, T is onto. [Range of T is all of V iff T is onto]

Conversely, Assume T is onto.

To prove: T is regular

Suppose not, T is singular.

write the Theorem

By Theorem: 6.1.3, there exists $v_1 \neq 0$ in V such

Since $v_1 \neq 0$.

Then by Lemma: 4.2.5, we can find $v_2, v_3, \ldots, v_n$

Such that $v_1, v_2, \ldots, v_n$ form a basis of V.

Then every element in VT is a linear combination of $w_1 = v_1 T$, $w_2 = v_2 T, \ldots, w_n = v_n T$.

Here $w_1 = 0$, VT is spanned by n-1 elements $w_2, w_3, \ldots, w_n$.

$$\dim VT \leq n-1 < n = \dim V.$$

$\Rightarrow$ VT and V are not equal.

$\therefore$ T is not onto.

which is a contradiction.

Hence T is regular.

Note:

An element $T \in A(V)$ is regular iff $\dim(VT) = \dim V$.

Defn:

If V is finite-dimensional over F, then the rank of T is the dimension of VT, the range of T over F.

ie) rank of T = dimension of range of T.

We denote it by rank of T (or) $r(T)$.

Note:

(1) If $r(T) = \dim V$ then T is regular.

Since $r(T) = \dim(VT)$

Given $r(T) = \dim V$

$\Rightarrow \dim(VT) = \dim V$

$\Rightarrow$ T is regular.

(2) If $r(T) = 0$ then $T = 0$.

ie) T is singular.

J
2017
2011

Lemma: 6.1.3 If V is finite-dimensional over F, then for $S, T \in A(V)$.

1) $r(ST) \leq r(T)$

2) $r(TS) \leq r(T)$ (and so $r(ST) \leq \min\{r(T), r(S)\}$)
(and so $r(ST) \leq \min\{r(T), r(S)\}$)

3) $r(ST) = r(TS) = r(T)$ for S regular in $A(V)$.

Proof:

1) Let $S, T \in A(V)$.

Then $VS \subset V$

Now, $V(ST) = (VS)T \subset VT$

By Lemma: 4.2.6, $\dim(V(ST)) \leq \dim(VT)$

ie) $r(ST) \leq r(T)$ —— ①

2) Suppose that $r(T) = m$.

$\Rightarrow VT$ has a basis of m elements say $w_1, w_2, \ldots, w_n$.

$\Rightarrow (VT)S$ is spanned by $w_1S, w_2S, \ldots, w_nS$.

$\Rightarrow (VT)S$ has dimension atmost m.

$\Rightarrow \dim((VT)S) \leq m$.

Now, $r(TS) = \dim(V(TS))$

$= \dim((VT)S)$

$\leq m = r(T)$

$\therefore r(TS) \leq r(T)$ —— ②

From ①, ② $r(ST) \leq \min\{r(S), r(T)\}$

3) Let $S \in A(V)$ is regular.

Then by Theorem: 6.1.4, S maps V onto V.

ie) $VS = V$.

Now, $V(ST) = (VS)T = VT$

$\Rightarrow \dim(V(ST)) = \dim(VT)$

$\Rightarrow r(ST) = r(T)$

Let $r(T) = m \Rightarrow \dim(VT) = m$.

Let $w_1, w_2, \ldots, w_m$ be a basis of VT.

Then $w_1 S, w_2 S, \ldots, w_m S$ spanned by $(VT)S = V($

Claim: $w_1 S, w_2 S, \ldots, w_m S$ are linearly Independent.

Let $\alpha_1(w_1 S) + \alpha_2(w_2 S) + \cdots + \alpha_m(w_m S) = 0$.

$(\alpha_1 w_1)S + (\alpha_2 w_2)S + \cdots + (\alpha_m w_m)S = 0$.

$(\alpha_1 w_1 + \alpha_2 w_2 + \cdots + \alpha_m w_m)S = 0$

Since S is ~~Singular~~ regular, S^{-1} exists.

$\Rightarrow (\alpha_1 w_1 + \alpha_2 w_2 + \cdots + \alpha_m w_m)SS^{-1} = 0 \cdot S^{-1}$

$\alpha_1 w_1 + \alpha_2 w_2 + \cdots + \alpha_m w_m = 0$.

Since $w_1, w_2, \ldots, w_m$ are linearly independent

$\Rightarrow \alpha_1 = \alpha_2 = \cdots = \alpha_m = 0$.

Hence $w_1 S, w_2 S, \ldots, w_m S$ form a basis for $V(TS)$

Now, $r(TS) = \dim(V(TS)) = m = \dim(VT) = r($

$\therefore r(ST) = r(TS) = r(T)$.

Corollary:

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AP-2019

If $T \in A(V)$ and if $S \in A(V)$ is regular, Then $r(T) = r(STS^{-1})$

3 marks Proof

Proof:

Now, $r(STS^{-1}) = r(S(TS^{-1}))$

$= r((TS^{-1})S)$ [By Lemma: 6.1.3 (

$= \gamma(T(S^{-1}S))$

$= \gamma(TI)$

$= \gamma(T)$

$\therefore \gamma(STS^{-1}) = \gamma(T)$

Sec: 6.2 Characteristic roots

Defn: If $T \in A(V)$ then $\lambda \in F$ is called a characteristic root (or singular) of T if $\lambda - T$ is singular.

ie) $\lambda \cdot 1 - T$ is singular. where 1 represent the unit element in $A(V)$.

J 2012 Theorem: 6.2.1

The element $\lambda \in F$ is a characteristic root of $T \in A(V)$ iff for some $v \neq 0$ in V, $vT = \lambda v$

Proof:

Let $\lambda \in F$ be a characteristic root of T.

$\Rightarrow \lambda - T$ is singular.

By Theorem: 6.1.3, $\exists\ v \neq 0$ in V such that $v(\lambda - T) = 0$

proof write pannanum

$\Rightarrow v(\lambda \cdot 1 - T) = 0$

$\Rightarrow v(\lambda \cdot 1) - v \cdot T = 0$

$\Rightarrow \lambda(v) - vT = 0$

$\Rightarrow \lambda v = vT$.

Conversely,

suppose $v \neq 0$ such that $vT = \lambda v$

$\Rightarrow v(\lambda - T) = 0$.

Thus $\exists\ v \neq 0$ such that $v(\lambda - T) = 0$.

By Theorem: 6.1.3, $\lambda - T$ is singular.

$\therefore \lambda$ is a characteristic root of T.

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2011, 2009, 2017, 2014, Ap 2019

Lemma: 6.2.1

If $\lambda \in F$ is a characteristic root of $T \in A(V)$ then PT for any polynomial $q(x) \in F[x]$, $q(\lambda)$ is a characteristic root of $q(T)$.

Proof:

Suppose that $\lambda \in F$ is a characteristic root of T

By the above Theorem,

Proof write Parnanum

$\exists\ v \neq 0$ such that $vT = \lambda v$.

Now, $vT^2 = (vT)T$
$= (\lambda v)T$
$= \lambda(vT)$
$= \lambda(\lambda v)$
$= \lambda^2 v.$

$vT^3 = (vT^2)T$
$= (\lambda^2 v)T$
$= \lambda^2(vT)$
$= \lambda^2(\lambda v)$
$= \lambda^3 v$

Continuning this way we get,

$vT^k = \cancel{\lambda} = \lambda^k \cdot v \quad \forall$ positve integer k.

Let $q(x) = \alpha_0 x^m + \alpha_1 x^{m-1} + \dots + \alpha_m$ where $\alpha_i \in F$ be a polynomial in $F[x]$.

Now, for any $v \neq 0$.

$v(q(T)) = v(\alpha_0 T^m + \alpha_1 T^{m-1} + \dots + \alpha_m)$
$= v(\alpha_0 T^m) + v(\alpha_1 T^{m-1}) + \dots + v(\alpha_m)$
$= \alpha_0 (vT^m) + \alpha_1 (vT^{m-1}) + \dots \ \alpha_m v$
$= \alpha_0 (\lambda^m v) + \alpha_1 (\lambda^{m-1} v) + \dots + \alpha_m v$

$= (\alpha_0 \lambda^m) v + (\alpha_1 \lambda^{m-1}) v + \cdots + \alpha_m v$

$= (\alpha_0 \lambda^m + \alpha_1 \lambda^{m-1} + \cdots + \alpha_m) v$

$= q(\lambda) v$

By Theorem: 6.2.1, The element $\lambda \in F$ is a characteristic root of $T \in A(V)$ iff for some $v \neq 0$ in V, $vT = \lambda v$ ~~such that $vT = \lambda v$~~

$q(\lambda)$ is a characteristic root of $q(T)$.

Theorem: 6.2.2

2014 J AP-2019

If $\lambda \in F$ is a characteristic root of $T \in A(V)$, then P.T λ is a root of the minimal polynomial of T. In particular, T only has a finite no. of characteristic root in F.

Proof:

Let λ be a characteristic root of T and $P(x)$ be a minimal polynomial over F of T.

Then $p(T) = 0$.

Since λ is a characteristic root of T.

By Theorem: 6.2.1, The element $\lambda \in F$ is a ch root of $T \in A(V)$ iff for some $v \neq 0$ in V, $vT = \lambda v$.

proof write pannanum

$\exists\, v \neq 0$ in V such that $vT = \lambda v$.

By Lemma: 6.2.~~1~~,

$P(\lambda)$ is a characteristic root of $P(T)$.

ie) $v(P(T)) = P(\lambda) v$

$\Rightarrow P(\lambda) v = 0 \quad (\because p(T) = 0)$

$\Rightarrow P(\lambda) = 0 . \quad (\because v \neq 0)$

$\therefore \lambda$ is a root of $P(x)$.

Since $\deg p(x) \leq n^2$ where $n = \dim_F V$ [By Theorem: 6.1.1]

$\Rightarrow P(x)$ has atmost n^2 roots.

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These finite no. of roots are also finite no. of Characteristic root of T in F.

Lemma: 6.2.2

If $T, S \in A(V)$ and if S is regular, then T and STS^{-1} have the same minimal polynomial.

Proof:

Let $T, S \in A(V)$ such that S is regular.

Let $P(x) = \alpha_0 + \alpha_1 x + \cdots + \alpha_m x^m$ be a minimal polynomial for T.

$$\Rightarrow P(T) = 0$$

$$\Rightarrow \alpha_0 + \alpha_1 T + \cdots + \alpha_m T^m = 0$$

Now, $(STS^{-1})^2 = (STS^{-1})(STS^{-1})$

$= ST(S^{-1}S)TS^{-1}$

$= ST^2S^{-1}$

Similiarly, $(STS^{-1})^3 = (STS^{-1})^2(STS^{-1})$

$= (ST^2S^{-1})(STS^{-1})$

$= ST^2(S^{-1}S)TS^{-1}$

$= ST^3S^{-1}$

Continuing like this way we get,

$$(STS^{-1})^n = ST^nS^{-1}$$

$P(STS^{-1}) = \alpha_0 + \alpha_1(STS^{-1}) + \alpha_2(STS^{-1})^2 + \cdots + \alpha_m(STS^{-1})^m$

$= S\alpha_0 S^{-1} + \alpha_1(STS^{-1}) + \alpha_2(ST^2S^{-1}) + \cdots + \alpha_m(ST^mS^{-1})$

$= S\alpha_0 S^{-1} + S(\alpha_1 T)S^{-1} + S(\alpha_2 T^2)S^{-1} + \cdots + S(\alpha_m T^m)S^{-1}$

$= S(\alpha_0 + \alpha_1 T + \cdots + \alpha_m T^m) S^{-1}$

$= S\, P(T) S^{-1}$

$= 0.$

Let $q(x)$ be any other polynomial of degree $< m$

Such that $q(STS^{-1}) = 0.$

$\Rightarrow S\, q(T) S^{-1} = 0.$

$\Rightarrow S^{-1} S\, q(T) S^{-1} S = 0$ ($\because S$ is regular)

$\Rightarrow q(T) = 0.$

which is contradiction to $P(x)$ is minimal polynomial for T.

Hence $P(x)$ is minimal polynomial for STS^{-1}

Hence T and STS^{-1} have the same minimal polynomial.

Defn: The element $v \neq 0 \in V$ is called a characteristic vector of T belonging to the characteristic root $\lambda \in F$ if $vT = \lambda v$.

Theorem: 6·2·3

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If $\lambda_1, \lambda_2, \ldots, \lambda_k$ in F are distinct characteristic roots of $T \in A(V)$ and if $v_1, v_2, \ldots, v_k$ are characteristic vectors of T belonging to $\lambda_1, \ldots, \lambda_k$ respectively, then $v_1, v_2, \ldots, v_k$ are linearly independent over F.

Proof:

If $k = 1$. Then v_1 is a characteristic vector of T belonging to characteristic root λ_1.

Then $v_1 \neq 0$ suchthat $v_1 T = \lambda_1 v_1$.

Here $v_1 \neq 0$, then $\{v_1\}$ is linearly independent over F.

Suppose $k > 1$ ~~such that~~ If $\{v_1, v_2, \ldots, v_k\}$ are linearly independent

2020.12.05 15:40

Then $\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_k v_k = 0$ where $\alpha_1, \alpha_2, \dots, \alpha_k \in F$

and not all of them are 0.

$\Rightarrow$ some nonzero coefficients as possible by suitably renumbering the vector the shortest relation is

$$\beta_1 v_1 + \beta_2 v_2 + \dots + \beta_j v_j = 0 \quad \text{--- (1)}$$

where $\beta_1 \neq 0, \beta_2 \neq 0, \dots, \beta_j \neq 0$.

$\Rightarrow (\beta_1 v_1 + \dots + \beta_j v_j) T = 0$

$\Rightarrow \beta_1 (v_1 T) + \beta_2 (v_2 T) + \dots + \beta_j (v_j T) = 0$.

Since $v_1, v_2, \dots, v_k$ are characteristic vectors of T belonging to the characteristic roots $\lambda_1, \lambda_2, \dots, \lambda_k$ respectively.

Then $\lambda_i v_i = v_i T$ where $i = 1$ to k.

$\therefore \beta_1 (\lambda_1 v_1) + \beta_2 (\lambda_2 v_2) + \dots + \beta_j (\lambda_j v_j) = 0$

$$\lambda_1 \beta_1 v_1 + \lambda_2 \beta_2 v_2 + \dots + \lambda_i \beta_j v_j = 0 \quad \text{--- (2)}$$

(2) $-$ (1) $\times \lambda_1 \Rightarrow (\lambda_2 - \lambda_1) \beta_2 v_2 + \dots + (\lambda_j - \lambda_1) \beta_j v_j = 0$

Since λ_j's are distinct, $\lambda_i - \lambda_1 \neq 0$ for $i > 1$.

Hence $(\lambda_i - \lambda_1) \beta_i \neq 0$

which is a contradiction to (1) is shortest relation

Hence $\{v_1, v_2, \dots, v_k\}$ are linearly independent over F

Corollary: 1

If $T \in A(V)$ and if $\dim_F V = n$ then T can have atmost n distinct characteristic roots in F

Proof:

Suppose T has m distinct characteristic roots in F

Then T has m distinct characteristic ~~roo~~ vectors.

By the above Theorem these m vectors are linearly in since $\dim_F V = n$ then $m \leq n$.

ie) the no. of characteristic roots of T is atmost n.

Corollary: 2

If $T \in A(V)$ and if $\dim_F V = n$ and if T has n distinct characteristic roots in F then there is a basis of V over F which consists of characteristic vector of T.

Proof: Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the distinct characteristic roots of T.

Let $v_1, v_2, \ldots, v_n$ be the characteristic vector of T belonging to the characteristic root $\lambda_1, \lambda_2, \ldots, \lambda_n$ respectively.

To prove: $v_1, v_2, \ldots, v_n$ be a basis of V over F.

By the above Theorem, $\{v_1, v_2, \ldots, v_n\}$ are linearly independent over F.

To prove: $\{v_1, v_2, \ldots, v_n\}$ span V.

Let $v \in V$, since $\dim_F V = n$ then any set of $n+1$ elements in V are linearly independent.

$\Rightarrow$ The vectors $v, v_1, v_2, \ldots, v_n$ are linearly dependent over F.

$\Rightarrow$ v can be expressed as the linear combination of $v_1, v_2, \ldots, v_n$.

$\Rightarrow$ $v_1, v_2, \ldots, v_n$ span V.

Hence $\{v_1, v_2, \ldots, v_n\}$ form a basis of V.

Thus there is a basis of V over F which consists of characteristic vectors of T.

2020.12.05 15:41

Sec : 6.3 Matrices

Defn: Let V be an n-dimensional vector space over F and let $v_1, v_2, \ldots, v_n$ be a basis for V over F. If $T \in A(V)$ Then the matrix of T in the basis $v_1, v_2, \ldots, v_n$ defined by

$$m(T) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \vdots & \vdots & & \vdots \\ \alpha_{n1} & \alpha_{n2} & \cdots & \alpha_{nn} \end{pmatrix}$$

where $v_i T = \sum_{j=1}^{n} \alpha_{ij} v_j$ for $i = 1, 2, \ldots, n$, $\alpha_{ij} \in F$.

Example:

Let F be a field and V be the set of all polynomials in x of degree $n-1$ or less over F on V. let D be defined by

$$D(\beta_0 + \beta_1 x + \cdots + \beta_{n-1} x^{n-1}) = \beta_1 + 2\beta_2 x + \cdots + i\beta_i x^{i-1} + \cdots + (n-1)\beta_{n-1} x^{n-2}$$

Then D is a linear transformation on V.

1) Consider the basis $1, x, x^2, \ldots, x^{n-1}$

To find in $m(D)$ in the basis $v_1 = 1, v_2 = x, \ldots, v_n = x^{n-1}$

Soln:

Now, $v_1 D = 0 = 0v_1 + 0v_2 + \cdots + 0v_n$.

$v_2 D = 1 = 1v_1 + 0v_2 + \cdots + 0v_n$.

$v_3 D = 2x = 0.v_1 + 2v_2 + \cdots + 0v_n$.

$\vdots$

$v_n D = (n-1)x^{n-2} = 0v_1 + 0v_2 + \cdots + (n-1)v_{n-1} + 0v_n$.

The matrix of D in the basis $\{1, x, x^2, \ldots, x^n\}$ is

$$m(D) = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 3 & \cdots & 0 & 0 \\ \vdots & & & & & \\ 0 & 0 & 0 & \cdots & n-1 & 0 \end{pmatrix}$$

2) Consider the basis $w_1 = x^{n-1}$, $w_2 = x^{n-2}$, ..., $w_i = x^{n-i}$,

..... $w_{n-1} = x$, $w_n = 1$

To find the matrix of these basis.

Now, $w_1 D = (n-1)x^{n-2} = 0w_1 + (n-1)w_2 + 0w_3 + \cdots + 0w_n$

$w_2 D = (n-2)x^{n-3} = 0w_1 + 0w_2 + (n-2)w_3 + 0w_4 + \cdots + 0w_n$

$\vdots$

$w_i D = (n-i)x^{n-i-1} = 0w_1 + \cdots + 0w_i + (n-i)w_{i+1} + 0w_{i+2} + \cdots + 0w_n$

$\vdots$

$w_{n-1} D = 1 = 0w_1 + 0w_2 + \cdots + 1w_n$

$w_n D = 0 = 0w_1 + 0w_2 + \cdots + 0w_n$

$\therefore$ The matrix of D in the basis $\{x^{n-1}, x^{n-2}, \ldots, 1\}$ is

$$m(D) = \begin{pmatrix} 0 & (n-1) & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & (n-2) & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & (n-3) & \cdots & 0 & 0 \\ \vdots & & & & & & \\ 0 & 0 & 0 & \cdots & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & \cdots & 0 & 0 \end{pmatrix}$$

3) Consider the basis $u_1 = 1$, $u_2 = 1+x$, $u_3 = 1+x^2$, ..., $u_n = 1+x^{n-1}$

Find the matrix of these basis

Soln:

$u_1 D = 0 = 0u_1 + 0u_2 + \cdots + 0u_n$

$u_2 D = 1 = 1 \times u_1 + 0u_2 + \cdots + 0u_n$

$u_3 D = 2x = 2(u_2 - u_1) = -2u_1 + 2u_2 + 0u_3 + \cdots + 0u_n$

$u_4 D = 3x^2 = 3(u_3 - u_1) = -3u_1 + 3u_3$

$= -3u_1 + 0u_2 + 3u_3 + 0u_4 + \cdots + 0u_n$

....

....

$u_n D = (n-1)x^{n-2} = (n-1)(u_{n-1} - u_1) = -(n-1)u_1 + (n-1)u_{n-1}$

$= -(n-1)u_1 + 0u_2 + 0u_3 + \cdots + (n-1)u_{n-1} + 0u_n$

The matrix of D in this basis $\{1, 1+x, 1+x^2, \ldots, 1+x^{n-1}\}$

$$m(D) = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ -2 & 2 & 0 & \cdots & 0 & 0 \\ -3 & 0 & 3 & \cdots & 0 & 0 \\ \vdots & & & & & \\ -(n-1) & 0 & 0 & \cdots & (n-1) & 0 \end{pmatrix}$$

Note :

1) If $T \in A(V)$ where V is an n-dimensional vector space over F and if T has n distinct characteristic roots $\lambda_1, \lambda_2, \ldots, \lambda_n$ in F then by corollary: 2 to Theorem: 6.23 we find a basis $v_1, v_2, \ldots, v_n$ of V over F such that $v_i T = \lambda_i v_i$, $i = 1$ to n.

In this basis the matrix of T is

$$m(T) = \begin{pmatrix} \lambda_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 & 0 \\ \vdots & & & & & \\ 0 & 0 & 0 & \cdots & 0 & \lambda_n \end{pmatrix}$$

2) Once a basis $v_1, v_2, \ldots, v_n$ of V is given then every linear transformation T we can associate a matrix

Conversely, having choosen a basis $v_1, v_2, \ldots, v_n$ of over F then given matrix.

$$\begin{pmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \vdots & & & \\ \alpha_{n1} & \alpha_{n2} & \cdots & \alpha_{nn} \end{pmatrix} \quad \text{--- (1)}$$

, with $\alpha_{ij} \in F$ gives rise to a linear transformation T defined on V by $v_i T = \sum_{j=1}^{n} \alpha_{ij} v_j$ on this basis. Thus every possible square array serves as matrix of some linear transformation in this basis $v_1, v_2, \ldots, v_n$.

In the matrix (1) the element α_{ij} is the intersection of the i^{th} row and j^{th} column.

ie) $(i, j)^{th}$ entry of the matrix.

If V is a n-dimensional vector space over F and $v_1, v_2, \ldots, v_n$ is a basis of V over F and S, T are linear transformation on V over F having matrices $m(S) = (\sigma_{ij})$ and $m(T) = (\tau_{ij})$ respectively in the given basis then

i) $m(S) = m(T)$ iff $\sigma_{ij} = \tau_{ij}$

ii) $m(S+T) = \lambda_{ij}$ where $\lambda_{ij} = \sigma_{ij} + \tau_{ij}$

iii) $m(\lambda S) = \mu_{ij}$ where $\mu_{ij} = \lambda \sigma_{ij}$

iv) $m(ST) = \vartheta_{ij}$ where $\vartheta_{ij} = \sum_{k=1}^{n} \sigma_{ik} \tau_{kj}$

Note:

1) $m(S+T) = m(S) + m(T)$

2) $m(\lambda S) = \lambda m(S)$

3) $m(ST) = m(S)\, m(T)$.

Example:

If $m(S) = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and $m(T) = \begin{pmatrix} -1 & 0 \\ 2 & 3 \end{pmatrix}$ then find $m(ST) = ?$

Here $\sigma_{11} = 1$, $\sigma_{12} = 2$, $\sigma_{21} = 3$, $\sigma_{22} = 4$ and

$\tau_{11} = -1$, $\tau_{12} = 0$, $\tau_{21} = 2$, $\tau_{22} = 3$.

using (iv), $\vartheta_{11} = \sum_{k=1}^{2} \sigma_{1k} \tau_{k1} = \sigma_{11}\tau_{11} + \sigma_{12}\tau_{21}$

$= (1)(-1) + (2)(2) = 3.$

Similiarly, $\vartheta_{12} = \sum_{k=1}^{2} \sigma_{1k} \tau_{k2} = \sigma_{11}\tau_{12} + \sigma_{12}\tau_{22}$

$= (1)(0) + (2)(3) = 6.$

$\vartheta_{21} = \sum_{k=1}^{2} \sigma_{2k} \tau_{k1} = \sigma_{21}\tau_{11} + \sigma_{22}\tau_{21}$

$= (3)(-1) + (4)(2) = -3+8 = 5.$

$\vartheta_{22} = \sum_{k=1}^{2} \sigma_{2k} \tau_{k2} = \sigma_{21}\tau_{12} + \sigma_{22}\tau_{22}$

$= (3)(0) + (4)(3) = 12.$

$\therefore m(ST) = \begin{pmatrix} 3 & 6 \\ 5 & 12 \end{pmatrix}$

2020.12.05

$$\therefore m(S).m(T) = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} -1+4 & 0+6 \\ -3+8 & 0+12 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 6 \\ 5 & 12 \end{pmatrix}$$

$$\therefore m(ST) = m(S).m(T).$$

Note: Let F_n = set of all $n \times n$ matrix whose entries are in F

$$= \left\{ \begin{pmatrix} \alpha_{11} & \alpha_{12} & \dots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \dots & \alpha_{2n} \\ \vdots & & & \\ \alpha_{n1} & \alpha_{n2} & \dots & \alpha_{nn} \end{pmatrix} \middle/ \alpha_{ij} \in F \right\}$$

$$= \{ (\alpha_{ij}) / \alpha_{ij} \in F \}$$

By note, In F_n.

i) $(\alpha_{ij}) = (\beta_{ij})$ iff $\alpha_{ij} = \beta_{ij}$ $\forall$ i and j

ii) $(\alpha_{ij}) + (\beta_{ij}) = \lambda_{ij}$ where $\lambda_{ij} = \alpha_{ij} + \beta_{ij}$ $\forall$ i and j

iii) $\gamma(\alpha_{ij}) = \mu_{ij}$ where $\mu_{ij} = \gamma\alpha_{ij}$ $\forall$ i and j

iv) $(\alpha_{ij})(\beta_{ij}) = \gamma_{ij}$ where $\gamma_{ij} = \sum_{k=1}^{n} \alpha_{ik}\beta_{kj}$ $\forall$ i and j

Theorem: 6.3.1

The set of all $n \times n$ matrices over F form a associative algebra, F_n, over F. Then A(V) and F_n are isomorphic as algebra over F If V is an n-dimensional vector space over F, then A(V) and F_n are isomorphic as algebras over F. Given any basis $v_1, \dots, v_n$ of V over F, if for $T \in A(V)$, $m(T)$ is the matrix of T in the basis $v_1, \dots, v_n$, the mapping $T \to m(T)$ provides an algebra isomorphism of A(V) onto F_n

Proof: Let $F_n = \{ (\alpha_{ij}) / \alpha_{ij} \in F \}$ be the set of all $n \times n$ matrices over F.

In F_n, define addition, scalar multiplication and also multiplication are given by.

$(\alpha_{ij}) + (\beta_{ij}) = (\lambda_{ij})$ where $\lambda_{ij} = \alpha_{ij} + \beta_{ij}$

$\lambda(\alpha_{ij}) = (\mu_{ij})$ where $\mu_{ij} = \lambda\alpha_{ij}$

$(\alpha_{ij})(\beta_{ij}) = (\gamma_{ij})$ where $\gamma_{ij} = \sum_{k=1}^{n} \alpha_{ik}\beta_{kj}$

To prove: F_n is associative algebra

It is an abelian group under "+"

Define $\boxed{\psi : A(V) \to F_n}$ by $\psi(T) = m(T)$ the matrix of T in the basis $v_1, v_2, \ldots, v_n$

Now, $\psi(S+T) = m(S+T)$

$= m(S) + m(T)$

$= \psi(S) + \psi(T)$

$\psi(\lambda S) = m(\lambda S)$

$= \lambda\, m(S)$

$= \lambda \psi(S)$

$\psi(ST) = m(ST)$

$= m(S) \cdot m(T)$

$= \psi(S) \cdot \psi(T)$

To prove: ψ is 1-1

Let $S, T \in A(V)$ such that $\psi(S) = \psi(T)$

$\Rightarrow m(S) = m(T)$

$\Rightarrow S = T$

$\therefore \psi$ is 1-1

To prove: ψ is onto

Then the linear transformation $S : V \to V$ defined by $v_i S = \sum_{j=1}^{n} \alpha_{ij} v_j$.

$m(S) = (\alpha_{ij})$

ie) $\psi(S) = (\alpha_{ij})$

$\therefore \psi$ is onto.

Hence ψ is an algebra isomorphism of $A(V)$ onto F_n

ie) $A(V) \cong F_n$ is algebra over F.

Theorem: 6.3.2

If V is n-dimensional over F and if $T \in A(V)$ has the matrix $m_1(T)$ in the basis $v_1, \dots, v_n$ and the matrix $m_2(T)$ in the basis $w_1, \dots, w_n$ of V over F, then there is an element $C \in F_n$ such that $m_2(T) = C m_1(T) C^{-1}$. In fact, if S is the linear transformation of V defined by $v_i S = w_i$ for $i = 1, 2, \dots n$, then C can be chosen to be $m_1(S)$.

Proof:

Let $m_1(T) = (\alpha_{ij})$ and $m_2(T) = (\beta_{ij})$

Then $v_i T = \sum_{j=1}^{n} \alpha_{ij} v_j$ and $w_i T = \sum_{j=1}^{n} \beta_{ij} w_j$

Let S be the linear transformation on V defined by $v_i S = w_i$, $i = 1$ to n.

Since $\{v_1, \dots, v_n\}$ and $\{w_1, \dots, w_n\}$ are the basis of V over F. Then S maps V onto V.

$\Rightarrow S$ is invertible in $A(V)$ [By Theorem: 6.1.4]

Now, $w_i T = \sum_{j=1}^{n} \beta_{ij} w_j$ —— (1)

Since $w_i = v_i S$

Then. $(1) \Rightarrow (v_i S)T = \sum_{j=1}^{n} \beta_{ij} v_j S$

$$v_i(ST) = \sum_{j=1}^{n} \beta_{ij} v_j S$$

Since S is invertible, S^{-1} exists

$$\therefore v_i(ST)S^{-1} = \sum_{j=1}^{n} \beta_{ij} v_j SS^{-1}$$

$$v_i(STS^{-1}) = \sum_{j=1}^{n} \beta_{ij} v_j$$

$$\therefore m_1(STS^{-1}) = (\beta_{ij}) = m_2(T)$$

We know that, $T \to m_1(T)$ is an isomorphism of $A(V)$ onto F_n.

We have $m_1(STS^{-1}) = m_1(S)\, m_1(T)\, m_1(S^{-1})$

$$= m_1(S)\, m_1(T) (m_1(S))^{-1}$$

Take $C = m_1(S)$

Hence $m_1(STS^{-1}) = C\, m_1(T)\, C^{-1}$.

Example:

Let V be the vector space of polynomial over F of degree ≤ 3. Let D be the differentiation operator defined by $(\alpha_0 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3) D = \alpha_1 + 2\alpha_2 x + 3\alpha_3 x^2$.

Consider the basis $v_1 = 1$, $v_2 = x$, $v_3 = x^2$, $v_4 = x^3$

To find: $m_1(D)$ is

Now, $v_1 D = 0 = 0v_1 + 0v_2 + 0v_3 + 0v_4$

$$v_2 D = 1 = 1 \cdot v_1 + 0 \cdot v_2 + 0v_3 + 0v_4$$

$$v_3 D = 2x = 0v_1 + 2v_2 + 0v_3 + 0v_4$$

$$v_4 D = 3x^2 = 0v_1 + 0v_2 + 3v_3 + 0v_4.$$

$$\therefore m_1(D) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \end{pmatrix}$$

Consider the basis $u_1 = 1$, $u_2 = 1+x$, $u_3 = 1+x^2$, $u_4 = 1+x^3$.

Now, $u_1 D = 0 = 0u_1 + 0u_2 + 0u_3 + 0u_4$.

$u_2 D = 1 = 1u_1 + 0u_2 + 0u_3 + 0u_4$

$u_3 D = 2x = 2(u_2 - u_1) = -2u_1 + 2u_2 + 0u_3 + 0u_4$

$u_4 D = 3x^2 = -3(u_3 - u_1) = -3u_1 + 0u_2 + 3u_3 + 0u_4$

$$\therefore m_2(D) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ -2 & 2 & 0 & 0 \\ -3 & 0 & 3 & 0 \end{pmatrix}$$

Let S be the linear transformation of V defined by $v_i S = u_i \ \forall\, i = 1,2,3,4$.

Then $v_1 S = u_1 = 1 = v_1$

$v_2 S = u_2 = 1+x = v_1 + v_2$

$v_3 S = u_3 = 1+x^2 = v_1 + v_3$.

$v_4 S = u_4 = 1+x^3 = v_1 + v_4$.

$\therefore$ The matrix of S in the basis v_1, v_2, v_3, v_4 is

$$m_1(S) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

Take $C = m_1(S)$

$$C^{-1} = \frac{1}{|C|} \operatorname{adj} C.$$

$$|C| = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{vmatrix} = 1 \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$|C| = 1(1-0) = 1.$$

Cofactor of $a_{11} = + \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1.$

Cofactor of $a_{12} = - \begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix} = -1$

Cofactor of $a_{13} = + \begin{vmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{vmatrix} = 1(0-0) - 1(1-0)$
$= -1$

Cofactor of $a_{14} = - \begin{vmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{vmatrix} = -[1(0-0) - 1(0-1)]$
$= -1$

Cofactor of $a_{21} = - \begin{vmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 0.$

Cofactor of $a_{22} = + \begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix} = 1(1-0) = 1.$

Cofactor of $a_{23} = - \begin{vmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{vmatrix} = -[1(0-0)] = 0$

Cofactor of $a_{24} = + \begin{vmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{vmatrix} = 1(0-0) = 0.$

Cofactor of $a_{31} = \begin{vmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 0.$

Cofactor of $a_{32} = -\begin{vmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{vmatrix} = 1(0-0) = 0.$

Cofactor of $a_{33} = +\begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix} = 1(1-0)+0 = 1$

Cofactor of $a_{34} = -\begin{vmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{vmatrix} = 1(0-0) = 0.$

Cofactor of $a_{41} = -\begin{vmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = 0.$

Cofactor of $a_{42} = +\begin{vmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{vmatrix} = 1(0-0) = 0.$

Cofactor of $a_{43} = -\begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix} = 1(0-0) = 0.$

Cofactor of $a_{44} = +\begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix} = 1(1-0) = 1.$

$$\text{adj } C = \begin{pmatrix} 1 & -1 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}^T$$

$$C^{-1} = \frac{1}{1}\begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}$$

2020.12.05

$$C^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}$$

Now,

$$C \cdot m_1(D) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \end{pmatrix}$$

$$C \cdot m_1(D)\, C^{-1} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ -2 & 2 & 0 & 0 \\ -3 & 0 & 3 & 0 \end{pmatrix}$$

$$= m_2(D)$$

$$\therefore\ C\, m_1(D)\, C^{-1} = m_2(D).$$